and then using the peak corresponding to some assumed profile provides a straightforward starting point. The use of a variable frequency microwave source at a "magnetoplasma resonance cutoff" condition ($\beta_- \to 0$) appears reasonable at least for the electron density conditions given in Table 2 of Ref. 3. Although significant changes in the magnetic field may alter the electron density profile, a 5% field variation should be sufficient for data acquisition.

Finally, spatial variations in the magnetic field as well as those in the electron density as mentioned earlier will influence the data reduction, although known field variations could be included in the integration of Eq. (1).

References

[1]Self, S. A., Reigel, F. O., Clements, R. M., and James, R. K., "Electron Concentration Measurements in Combustion MHD Flows by Submillimeter Laser Interferometry," *Journal of Energy*, Vol. 1, July-Aug. 1977, pp. 206-211.

[2]Annen, K. D., Kuzmenko, P. J., Keating, R., and Self, S. A., "Electron and Positive Ion Measurements in MHD Combustion Plasmas with Phosphorous Addition," *Journal of Energy*, Vol. 5, Jan.-Feb. 1981, pp. 31-38.

[3]Kuzmenko, P. J. and Self, S. A., "Measurement of Plasma Conductivity Using Faraday Rotation of Submillimeter Waves," *Journal of Energy*, Vol. 7, March-April 1983, pp. 176-181.

[4]Wharton, C. B. and Slager, D. M., "Microwave Determination of Plasma Density Profiles," *Journal of Applied Physics*, Vol. 31, 1960, pp. 428-430.

[5]Heald, M. A. and Wharton, C. B., *Plasma Diagnostics with Microwaves*, John Wiley & Sons, New York, 1965.

[6]Warder, R. C. Jr., "Microwave Measurements of Arc-Heated Argon Plasmas," Northwestern University, Evanston, Ill., NU-GDL Rept. B-1-63, April 1963.

Transonic Flow in the Throat Region of Radial or Nearly Radial Supersonic Nozzles

B. F. Carroll* and J. C. Dutton†
Texas A&M University, College Station, Texas

Introduction

In a recent paper, Conley et al.[1] analyzed annular nozzles using a time-dependent transonic technique to provide a starting line for supersonic method-of-characteristics calculations. Their attempts to analyze geometries for which the radial velocity components were large were hampered by the lack of an adequate transonic solution for the case of highly inclined throat flow. According to Ref. 1, "This inadequacy in predictive capability is identified as a major deficiency in the state-of-the-art of nozzle flowfield analysis." In response to this need, a transonic solution is presented that is applicable to annular nozzles with the throat flow inclined at arbitrary, but large, angles to the nozzle axis of symmetry. This solution is a computationally efficient means of establishing a supersonic initial value line for space-marching supersonic calculations. Both radial inflow and outflow as well as planar geometries may be analyzed with this solution.

Received Aug. 6, 1984.

*ONR Graduate Fellow, Department of Mechanical Engineering. Student Member AIAA.

†Associate Professor, Department of Mechanical Engineering. Member AIAA.

In previous work, Dutton and Addy presented third-order series expansion solutions for transonic flow in axisymmetric nozzles[2] and annular nozzles[3] that showed excellent agreement with experimental measurements. For annular nozzles, the perturbation velocity components were expanded in power series in the parameter ϵ, defined as $\epsilon \equiv 1/(\bar{R}_c + \eta)$, where $\bar{R}_c$ is an average dimensionless wall radius of curvature at the throat and η an arbitrary parameter included to improve the convergence of the series. The inclusion of the parameter η was shown to extend the applicability of the series solution to nozzles with small wall radii of curvature at the throat. However, only annular nozzles with small flow inclination angles could be analyzed, as a result of the order-of-magnitude assumptions made in the problem formulation.

Problem Formulation and Solution

A regular perturbation technique is to be used to obtain an approximate solution to the throat flowfield sketched in Fig. 1. The inclination angle β is measured between the Z axis of symmetry and the main flow x direction and is positive in the counterclockwise direction. The governing equations are taken as the irrotationality condition and the gasdynamic equation; the boundary conditions are that the nozzle walls are streamlines. These equations are transformed from the cylindrical R-Z coordinate system to the local dimensionless x-y system with lengths nondimensionalized by the throat half-height D and velocities by the critical speed of sound a^*. The perturbation velocity components $\tilde{u}$ and $\tilde{v}$ are defined by $u = 1 + \tilde{u}$ and $v = \tilde{v}$. Using the expansion parameter defined by Dutton and Addy[3] and discussed in the introduction, strict order-of-magnitude estimates[4] show that $\tilde{u} = \mathcal{O}(\epsilon)$, $\tilde{v} = \mathcal{O}(\epsilon^{3/2})$, and $x = \mathcal{O}(\epsilon^{1/2})$. In this analysis, $\sin\beta$ was taken as $\mathcal{O}(1)$ and $\cos\beta$ as $\mathcal{O}(\epsilon^{1/2})$, i.e., the radial or nearly radial configuration, $\beta \cong \pm\pi/2$.

Defining a new $\mathcal{O}(1)$ stretched axial coordinate z as

$$z \equiv \left(\frac{2}{\gamma + 1}\right)^{1/2} x\epsilon^{-1/2} \tag{1}$$

the appropriate expansions for $\tilde{u}$ and $\tilde{v}$ are

$$\tilde{u}(z,y) = u_1(z,y)\epsilon + u_2(z,y)\epsilon^2 + u_3(z,y)\epsilon^3 + \ldots \tag{2}$$

$$\tilde{v}(z,y) = \left(\frac{\gamma+1}{2}\epsilon\right)^{1/2} [v_1(z,y)\epsilon + v_2(z,y)\epsilon^2 + v_3(z,y)\epsilon^3 + \ldots] \tag{3}$$

where γ is the gas specific heat ratio. Using the order-of-magnitude estimates,[4] all variables are redefined in terms of $\mathcal{O}(1)$ quantities. Equations (2) and (3) along with the $\mathcal{O}(1)$ quantities are then substituted into the governing equations and boundary conditions. The boundary conditions must first be expanded in Taylor series about $y = \pm 1$ and the equations defining the wall contours are also expanded in Maclaurin series about $x = 0$. Coefficients of like powers of ϵ are gathered and equated, resulting in the following formulation for the various solution orders. The irrotationality con-

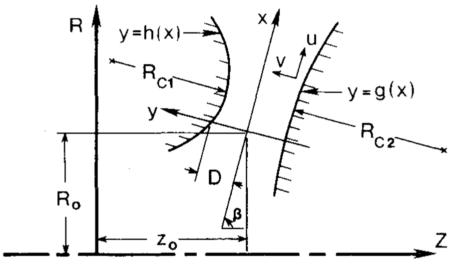

Fig. 1 Schematic of throat flowfield in a radial supersonic nozzle.

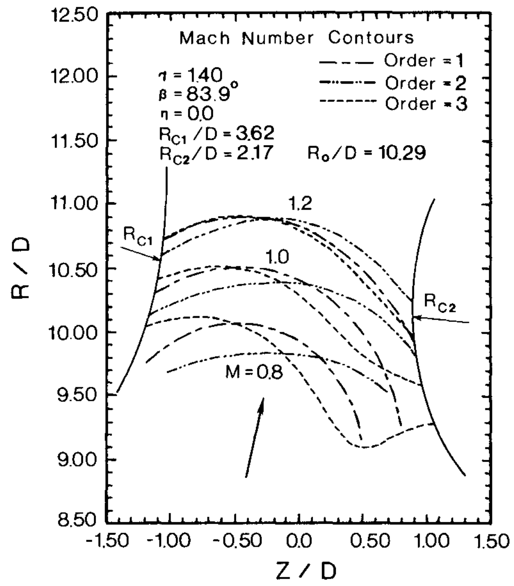

Fig. 2 Mach number contours for typical radial outflow configuration, $\eta=0$.

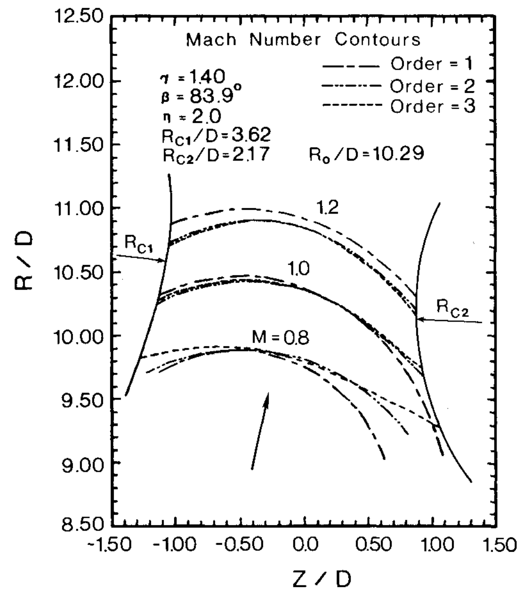

Fig. 3 Mach number contours for typical radial outflow configuration, $\eta=2$.

dition is given by

$$\frac{\partial u_n}{\partial y}-\frac{\partial v_n}{\partial z}=0 \quad (n=1,2,3,\ldots) \tag{4}$$

and the gasdynamic equation is given by

$$-2u_1\frac{\partial u_1}{\partial z}+\frac{\partial v_1}{\partial y}+S_1=0 \quad (n=1) \tag{5}$$

$$-2u_1\frac{\partial u_n}{\partial z}-2u_n\frac{\partial u_1}{\partial z}+\frac{\partial v_n}{\partial y}=f_n(u_1,v_1,\ldots,u_{n-1},v_{n-1}) \quad (n=2,3,\ldots) \tag{6}$$

where the functions f_n on the right-hand side of Eq. (6) are always known from the lower-order solutions and S_1 is a constant proportional to $\sin\beta$. The boundary conditions are given by

$$v_1(z,1)=h_1+h_2z \tag{7}$$

$$v_2(z,1)=(h_1+h_2z)u_1(z,1)+h_2\eta z \tag{8}$$

$$v_3(z,1)=(h_1+h_2z)u_2(z,1)+h_2\eta z u_1(z,1)+h_2\eta^2z-\left(\frac{\gamma+1}{2}\right)(h_1z+\tfrac{1}{2}h_2z^2)\left.\frac{\partial v_1}{\partial y}\right|_{(z,1)} \tag{9}$$

for the $y=h(x)$ boundary with similar expressions for the $y=g(x)$ boundary. The quantities h_1 and h_2 are $\mathcal{O}(1)$ expressions for the first and second derivatives, respectively, of the $y=h(x)$ contour evaluated at the throat.

The solution of Eqs. (4-6), subject to boundary conditions (7-9), proceeds one order at a time starting with the first order. An assumed solution form, as suggested by the boundary conditions, is substituted into the governing equations, resulting in two ordinary differential equations that are easily solved by direct integration. The analysis has been carried through the third order. Once the perturbation velocity components (u_1,v_1), (u_2,v_2), and (u_3,v_3) have been determined, suitable expansions can be used to determine the u and v velocity components, Mach number M, ratio of local velocity to critical speed of sound M^*, static-to-stagnation pressure ratio P/P_0, flow inclination angle θ, and discharge coefficient C_D. Further details concerning the problem formulation and a listing of the series solution may be found in Ref. 4.

Once obtained, the solution was thoroughly checked by several methods. One technique was to numerically back substitute it into a finite difference form of the problem formulation [Eqs. (4-9)]. In all cases, the series were found to satisfy the finite difference approximation to the governing equations and boundary conditions. The solution also correctly reduces to the simpler planar symmetric case of Hall[5] for the special case of planar nozzles with $R_{c1}=R_{c2}$ and $\eta=0$.

Results and Discussion

A parametric study[4] using the computer program in Ref. 6 was performed to establish guidelines for use of the transonic analysis. The main variables affecting the solution were found to be the dimensionless wall radii of curvature R_{c1}/D and R_{c2}/D and the convergence parameter η. Increasing the value of η improves the convergence characteristics, especially for small R_c/D, where R_c/D is the smaller of the two dimensionless wall radii of curvature. However, increasing η also tends to decrease the satisfaction of the exact boundary conditions [Eqs. (7-9) are approximate] and the conservation of mass in the region outside the throat. These competing effects limit the value of η that may be employed. For general use, a third-order analysis using as small a value of η as possible to achieve satisfactory convergence should be employed. A value of $\eta=1$ or 2 is recommended when $R_c/D\lesssim 2.0$. For larger wall radii of curvature, $\eta=0$ or 1 is recommended. The solution should be applied only in the throat region. Values of R_c/D as low as 0.6 were considered in the parametric study, which appears to be near the lower limit for obtaining accurate results.

Example transonic calculations are shown in Figs. 2 and 3 for a typical outflow geometry. All geometric parameters are identical in these two figures with only the value of η being changed from $\eta=0$ in Fig. 2 to $\eta=2$ in Fig. 3. Mach number contours are shown for all three orders. In Fig. 2 the solution is seen to be divergent for $\eta=0$, due to the relatively small wall radii of curvature. By increasing the value of η to 2 as in Fig. 3, however, the convergence has been greatly improved. In particular, note the excellent convergence between the second- and third-order contours for $M=1.0$ and 1.2.

Another characteristic evident in Fig. 3 is that the low supersonic contours ($M=1.2$) are generally more convergent than are the high subsonic ones ($M=0.8$). This figure demonstrates the ability of the series solution to provide initial value data for space-marching method of characteristics or finite difference analyses of radial supersonic nozzles.

Acknowledgment

This research was supported in part by an Office of Naval Research Graduate Fellowship for B. F. Carroll.

References

[1]Conley, R. R., Hoffman, J. D., and Thompson, H. D., "An Analytical and Experimental Investigation of Annular Propulsive Nozzles," AIAA Paper 84-0282, Jan. 1984.

[2]Dutton, J. C. and Addy, A. L., "Transonic Flow in the Throat Region of Axisymmetric Nozzles," *AIAA Journal*, Vol. 19, June 1981, pp. 801-804.

[3]Dutton, J. C. and Addy, A. L., "Transonic Flow in the Throat Region of Annular Supersonic Nozzles," *AIAA Journal*, Vol. 20, Sept. 1982, pp. 1236-1243.

[4]Carroll, B. F. and Dutton, J. C., "Transonic Throat Flow in Radial or Nearly Radial Supersonic Nozzles," Texas A&M University, College Station, TEES Rept. FP-84-01, May 1984.

[5]Hall, I. M., "Transonic Flow in Two-Dimensional and Axially-Symmetric Nozzles," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. XV, Pt. 4, 1962, pp. 487-508.

[6]Carroll, B. F. and Dutton, J. C., "RATLE, A Computer Program for Radial Transonic Nozzles," Texas A&M University, College Station, TEES Rept. FP-84-91, June 1984.

On the Vortex Stretching Modification of the k-ϵ Turbulence Model: Radial Jets

Arthur Rubel*
Grumman Aerospace Corporation,
Bethpage, New York

Introduction

ALTHOUGH the k-ϵ turbulence model has had success predicting free shear flows, it and other two-equation models appear to require a different set of constants to match both plane and round jet growth rates.[1-4] For example, Launder et al.[1] modified the eddy-viscosity coefficient C_μ and the destruction of dissipation constant $C_{\epsilon 2}$ to achieve round jet agreement. In the far field this modification reduced to a rather large C_μ correction. Raiszadeh and Dwyer[5] showed that k-ϵ model results are quite sensitive to the dissipation equation model constants $C_{\epsilon 1}$ and $C_{\epsilon 2}$; therefore, it is not surprising that these have been the target of several turbulence modelers.[4,6]

Pope[4] has given a phenomenological argument for incorporating a vortex stretching invariant term in the dissipation transport equation to modify the source terms. Unlike other models,[1,6] the form of the resulting equation has general application to three-dimensional problems. Pope settled on a third constant, $C_{\epsilon 3}$, such that his results matched round jet growth rate data. Plane jet agreement is guaranteed since there is no vortex stretching.

Performance of this model on the calculation of radially spreading jets has not been examined, although this case provides an excellent test[7] of the model. The radial jet is axisymmetric, like the round jet, but its velocity decays in the manner of a plane jet. In this Note the invariant vortex stretching modification to the k-ϵ model is applied to the self-preserving radial jet and shown to be inadequate.

Analysis

The present study of free jets in stagnant surroundings, using a k-ϵ model, is based on a far-field similarity formulation.[8,9] Similarity variables may be defined by

$$\frac{u}{u_0}=U(\eta),\quad \frac{k}{u_0^2}=G(\eta),\quad \frac{\epsilon x}{u_0^3}=H(\eta),\quad \frac{v}{u_0}=V(\eta),$$

$$\eta=\frac{y}{C_\mu^{1/2}x} \tag{1}$$

$$\eta^m U=(\eta^m F)',\quad C_\mu^{-1/2}V=\eta U-2^{j-1}F \tag{2}$$

where u,v and x,y are the streamwise, transverse velocities and coordinates, respectively. At $y=0$, the velocity $u_0(x)$ follows an $x^{-(j+1)/2}$ behavior. For plane jets, $j=m=0$; for radial jets, $j=1$, $m=0$, and for round jets, $j=m=1$. Primes indicate differentiation with respect to the similarity variable, η. Use of the stream function $\eta^m F$ in Eqs. (1) and (2) ensures the conservation of mass; the remaining thin shear layer equations become

Momentum:

$$2^{j-1}FU+\frac{G^2}{H}U'=0 \tag{3}$$

Energy:

$$2^j UG+2^{j-1}FG'+\frac{\sigma_k^{-1}}{\eta^m}\left(\eta^m\frac{G^2}{H}G'\right)'+\frac{G^2}{H}U'^2-H=0 \tag{4}$$

Dissipation:

$$\frac{1}{2}(5+3j)UH+2^{j-1}FH'+\frac{\sigma_\epsilon^{-1}}{\eta^m}\left(\eta^m\frac{G^2}{H}H'\right)'+C_{\epsilon 1}GU'^2$$

$$-C_{\epsilon 2}\frac{H^2}{G}+\underline{j\bar{C}_{\epsilon 3}\frac{G^2}{H}U'^2F'}=0 \tag{5}$$

where the underlined term of Eq. (5) represents the vortex stretching modification of Pope. Model constants are given by

$$C_\mu=0.09,\quad \sigma_k=1,\quad \sigma_\epsilon=1.3,$$

$$C_{\epsilon 1}=1.44,\quad C_{\epsilon 2}=1.90,\quad \bar{C}_{\epsilon 3}=C_{\epsilon 3}/4C_\mu=2.194 \tag{6}$$

Boundary conditions for the system are

$$\eta=0:\quad F=0,\quad U=1,\quad G'=0,\quad H'=0 \tag{7a}$$

$$\eta=\eta_e:\quad G=0,\quad H=0 \tag{7b}$$

Table 1 Comparison of calculated and observed jet growth rates

Jet	k-ϵ model	$y_{1/2}/x$ Calculated	$y_{1/2}/x$ Data[3,10,11]
Plane	—	0.1080	0.10-0.11
Round	Original	0.1199	
	Modified	0.0858	0.086
Radial	Original	0.0951	0.096-0.11
	Modified	0.0400	

Received June 25, 1984.

*Head, Theoretical Aerodynamics Laboratory, R&D Center. Member AIAA.